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On word structure of the modular group over finite and real quadratic fields

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Abstract

Let Ω denote the projective line over the real quadratic field and δ denote the projective line over the finite field F_q with q elements. Coset diagrams for the orbits of the modular group G acting on Ω and δ give some interesting information. By using these diagrams we determine a condition for the existence of an orbit of G on Ω containing a circuit of a given type. If such a circuit exists, we find a condition under which the orbit contains a real quadratic irrational number α along with its algebraic conjugate $\bar{\alpha}$. As there are two projections from Ω to δ we are interested in the case when G acts on δ and we determine necessary and sufficient conditions for the existence of two orbits of G: one containing α along with $1/\alpha$ and the other containing α together with $1/\bar{\alpha}$.

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1. Introduction

We adopt standard group-theoretic notations as used in [9]. It is well-known (see e.g. [2] for a proof of this using coset diagrams) that the modular group is generated by the linear-fractional transformations $x: z \to -1/z$ and $y: z \to (z-1)/z$ which satisfy the relations $x^2 = y^3 = 1$. Let G denote the modular group.

If q is a power of a prime p then the projective line over a finite field F_q, which is $F_q \bigcup \{\infty\}$, is denoted by $\mathrm{PL}(F_q)$. By Ω we shall mean the projective line over the real quadratic field. Throughout this paper, let α denote a real quadratic irrational number $(a+\sqrt{n})/c$, where n is a non-square positive integer and a, $(a^2-n)/c$, c are relatively prime integers. We denote the algebraic conjugate $(a-\sqrt{n})/c$ of α by $\bar{\alpha}$.

There is a well-known relation between the action of G on $\mathbb{R}$ and continued fractions. There are many papers pertaining to the connection between geodesics on the modular surface and continued fractions which are, in particular, important in the theory of approximation to real numbers by rationals (see e.g. [1, 4]). Important

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results have been obtained using these ideas in a nicely written paper [8]. It has been also shown in [6] that classifying the real quadratic irrational numbers α into the orbits of G (with respect to α) is almost the same as classifying reduced indefinite binary quadratic forms. A good account on relationship between continued fractions and indefinite binary quadratic forms is also given in [3].

The coset diagram idea is very similar to looking at groups acting on graphs or trees. We have used these diagrams to investigate local–global relationship between real quadratic irrational numbers and the elements of G. In this paper we have replaced $\mathbb{R}$ by Ω and $\mathrm{PL}(F_q)$. Coset diagrams for the orbit of G acting on real quadratic fields have given some interesting information in [5]. It has been shown in [5] that for a fixed value of a non-square positive integer, there are only a finite number of *ambiguous numbers* α and that part of the coset diagram containing α forms a single circuit and it is the only circuit in the orbit of α. In this paper we have classified these circuits further. We are concerned here with the continued fractions or word structure of the elements of G which generate an ambiguous number and symmetry relations of the word in the group. A relationship of the action of the element of G on $\mathrm{PL}(F_q)$ as compared with an ambiguous number is studied. Specifically, we show that under certain condition the orbit contains α along with $\bar{\alpha}$. In the case when G acts on $\mathrm{PL}(F_q)$, we determine necessary and sufficient conditions for the existence of two orbits of G; one containing α along with $1/\alpha$ and the other containing α together with $1/\bar{\alpha}$.

A coset diagram depicts a permutation representation of G: the 3-cycles of y are denoted by three vertices of a triangle permuted anti-clockwise by y and two vertices which are interchanged by x are joined by an edge. Fixed points of x and y are denoted by heavy dots. For examples, one can refer to [7].

If $p = \{v_0, e_1, v_1, e_2, \ldots, e_k, v_k\}$ is an alternating sequence of vertices and edges of a coset diagram, then π is a path in the diagram if e_i joins v_{i-1} and v_i for each i and $e_i \neq e_j$ $(i \neq j)$. By a circuit, we shall mean a closed path of edges and triangles. If $n_1, n_2, \ldots, n_{2k}$ is a sequence of positive integers then by a circuit of the type $(n_1, n_2, \ldots, n_{2k})$, we shall mean the circuit in which n_1 triangles have one vertex outside the circuit and n_2 triangles have one vertex inside the circuit and so on. This circuit induces an element $g = (xy)^{n_1}(xy^{-1})^{n_2} \cdots (xy^{-1})^{n_{2k}}$ of G and fixes a particular vertex of a triangle lying on the circuit.

For example, by the circuit of the type (6, 2, 1, 3, 2, 1, 3, 2, 1, 2) we mean the circuit as shown in Fig. 1. This circuit induces an element $g = (xy)^6(xy^{-1})^2(xy)(xy^{-1})^3(xy)^2$ $(xy^{-1})(xy)^3(xy^{-1})^2(xy)(xy^{-1})^2$ of G which fixes the particular vertex k, shown in the diagram.

Consideration of the actions of G on Ω and $\mathrm{PL}(F_q)$ suggests the importance of circuits. Next, we shall answer the following questions:

Corresponding to a given sequence of positive integers $n_1, \ldots, n_{2k}$, if such a circuit exists, when does the orbit contain with α its conjugate $\bar{\alpha}$? If G acts on $\mathrm{PL}(F_q)$, when does there exist an orbit of G containing with α also the element $1/\alpha$ or $1/\bar{\alpha}$.

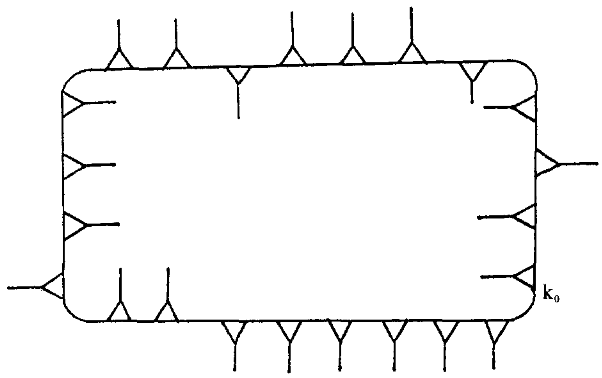

Fig. 1.

2. Action of G on Ω

When G acts on Ω, it is possible that $\alpha = (a + \sqrt{n})/c$ (where n is a non-square positive integer and $a, (a^2 - c)/c, c$ are relatively prime integers) and $\bar{\alpha}$ may have different signs. If such is the case then we call, as in [5], α an ambiguous number. In [5] it has been proved that for a fixed value of n there are only a finite number of ambiguous numbers and that the part of the coset diagram containing such numbers form a single circuit and it is the only one in the orbit of α. If k is the number of sets of triangles on the circuit, with one vertex outside the circuit and k' is the number of sets of triangles on the circuit with one vertex inside, then $k = k'$ and so the total number of sets of triangles in a circuit is $2k$. Further, we note that such sets of triangles, with one vertex outside/inside, occur alternately.

We will need the following theorem for our subsequent results.

Theorem 2.1. *Every element of G, except the (group-theoretic) conjugates of x, $y^{\pm 1}$ and $(xy)^n$, $n > 0$, has real quadratic irrational numbers as fixed points.*

Proof. Let $g: z \to (az + b)/(cz + d)$ belong to G and k_0 be a fixed point of g. Then $ck_0^2 + (d - a)k_0 - b = 0$ gives real roots only when $(d - a)^2 + 4bc \geqslant 0$. But $(d - a)^2 + 4bc = a^2 + d^2 - 2ad + 4bc = a^2 + d^2 - 2ad + 4ad - 4 = (a + d)^2 - 4 \geqslant 0$, where $a + d$ is the trace of the matrix corresponding to g. Now $(a + d)^2 - 4 \geqslant 0$ implies that if the roots are complex then $(a + d)^2 < 4$ and so $|a + d| = 0$ or ± 1. If $a + d = 0$ then g will take the form $z \to (az + b)/(cz - a)$ and so $g^2 = 1$. Since every element of G of order 2 is conjugate to x [9], hence g is a conjugate of $x: z \to -1/z$. Thus, the fixed points of conjugates of x are complex numbers. When $a + d = \pm 1$ we can replace a, b, c and d by $-a$, $-b$, $-c$ and $-d$ in g. So we can assume that $a + d = -1$. This means that the matrix

$$\begin{bmatrix} a & b \\ c & -a-1 \end{bmatrix}$$

for g, gives $M^2 + M + I = 0$ as its characteristic equation, which further yields $M^3 - I = 0$. This implies that $g: z \to (az + b)/(cz - (a + 1))$ has order 3 and so is a conjugate of $y: z \to (z - 1)/z$ because every element of G of order 3 is conjugate to y [9]. Hence, the fixed points of conjugates of $y^{\pm 1}$ are complex numbers.

If $a + d = \pm 2$, the characteristic equation in this case is $M^2 - 2M + I = 0$ and so by recursion (multiplying equation by M and substituting $2M - I$ for M^2) we get $M^n - nM + (n - 1)I = 0$. Thus g, in this case, is a conjugate of $(xy)^n: z \to z + n$ for some positive integer n and hence ∞ is the only fixed point of it.

If $a + d = \pm m > 2$, then $(a + d)^2 - 4 > 0$ and so the roots are real. In fact, $(a + d)^2 - 4$ cannot be a perfect square, for otherwise we shall be dealing with a coset diagram for rational numbers in which case we know from [5] that ∞ is the only fixed point. Thus, $(d + a)^2 - 4$ cannot be a perfect square and so the fixed points are real but irrational numbers. Thus, conjugates of $x, y^{\pm 1}, (xy)^n$, when $n > 0$, are the only exceptions where we do not get real quadratic irrational numbers as fixed points.

Let α be a real quadratic irrational number fixed by $g = (xy)^{n_1}(xy^{-1})^{n_2} \cdots (xy^{-1})^{n_{2k}}$, where $n_i > 0$ for all $i = 1, 2, \ldots, 2k$, of G. If $M(g)$ denotes the matrix corresponding to g, then the size of the trace of $M(g)$ determines the size of the circuit $(n_1, n_2, \ldots, n_{2k})$ containing, of course, α. This indeed means that there is a relationship between the sequence of positive integers $n_1, n_2, \ldots, n_{2k}$ and the trace of $M(g)$. In the following theorem we establish this relationship.

Theorem 2.2. *Let $W = \{1, 2, \ldots, 2k\}$ be the cyclically ordered set of positive integers and the orbit of α contain a circuit of the type $(n_1, n_2, \ldots, n_{2k})$, $n_i > 0$. Let P be the collection of non-empty subsets of W obtained by striking out any number of adjacent pairs of elements of W. Let $n_j = \prod_{i \in J} n_i$ for $J \in P$, then the trace of $M(g)$ is $2 + \sum_{J \in P} n_J$.*

Proof. Consider the element $g = (xy)^{n_1}(xy^{-1})^{n_2} \cdots (xy^{-1})^{n_{2k}}$, where n_i is a positive integer, of G corresponding to the circuit of the type $(n_1, n_2, \ldots, n_{2k})$ such that the real quadratic irrational number α is fixed by g. Since $xy: z \to z + 1$ and $xy^{-1}: z \to z/(z + 1)$ represent

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

therefore the matrix has the form

$$g(M) = \begin{bmatrix} 1 & n_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ n_2 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & n_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ n_j & 1 \end{bmatrix} \cdots \tag{1}$$

If we consider a matrix A written in the form

$$\begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} \end{bmatrix} \begin{bmatrix} a_{11}^{(2)} & a_{12}^{(2)} \\ a_{21}^{(2)} & a_{22}^{(2)} \end{bmatrix} \cdots \begin{bmatrix} a_{11}^{(m)} & a_{12}^{(m)} \\ a_{21}^{(m)} & a_{22}^{(m)} \end{bmatrix}$$

then the trace of A, of course, will be

$$\sum a^{(1)}_{\lambda_m\lambda_1} a^{(2)}_{\lambda_1\lambda_2} \cdots a^{(m)}_{\lambda_{m-1}\lambda_m} \tag{2}$$

In the case of $M(g)$ the factor matrices are alternately

$$\begin{bmatrix} 1 & n_i \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ n_j & 1 \end{bmatrix}$$

and therefore in any term of (2) if some $a^{(l)}_{\lambda_i\lambda_j} = 0$ then the entire term is 0. Moreover we can ignore any $a^{(l)}_{\lambda_i\lambda_j}$ which is 1. So, we shall consider only those terms which are neither 0 nor 1.

Let us consider the following portion of the three matrices occurring in (1):

$$\cdots \begin{bmatrix} 1 & 0 \\ n_p & 1 \end{bmatrix} \begin{bmatrix} 1 & n_q \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ n_r & 1 \end{bmatrix} \cdots \tag{3}$$

Since the trace of $M(g)$ is the sum of certain products obtained by choosing one entry from each matrix in (1), suppose that from the middle of the three matrices in (3) we choose n_q and in order to have a non-zero product we therefore shall choose n_r from the third matrix and 1 from the top left-hand corner of the first matrix. On the other hand, if from the first matrix we choose n_p instead of 1, then from the third matrix we need to choose 1 in the second row and the second column. In a similar way, if we consider the following portion of equation:

$$\cdots \begin{bmatrix} 1 & n_p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ n_q & 1 \end{bmatrix} \begin{bmatrix} 1 & n_r \\ 0 & 1 \end{bmatrix} \cdots \tag{4}$$

and choose n_q from the middle matrix we must then choose n_p from the first matrix and 1 in the first row and first column of the third matrix or instead if we choose 1 in the second row and second column of the first matrix then we need to choose n_r from the third matrix. In fact, this means that we are striking out any number of adjacent pairs of elements of W. Thus, for J in P if we let $n_J = \prod_{i\in J} n_i$ then the trace of $M(g)$ is $2 + \sum_{J\in P} n_J$.

For a given sequence of positive integers $n_1, n_2, \ldots, n_{2k}$ the circuit of the type

$$(n_1, n_2, \ldots, n_{2k'}, n_1, n_2, \ldots, n_{2k'}, \ldots, n_1, n_2, \ldots, n_{2k'}),$$

where k' divides k, is said to have a period of length $2k'$. In the following we prove that there does not exist a circuit of the above type in the orbit of G when acting on Ω.

Theorem 2.3. *For given positive integers $n_1, n_2, \ldots, n_{2k}$ there does not exist a circuit which has a period of length $2k'$, where k' divides k.*

Proof. If there exists a circuit which has a period of length $2k'$ then it will be of the type

$$(n_1, n_2, \ldots, n_{2k'}, n_1, n_2, \ldots, n_{2k'}, \ldots, n_1, n_2, \ldots, n_{2k'})$$

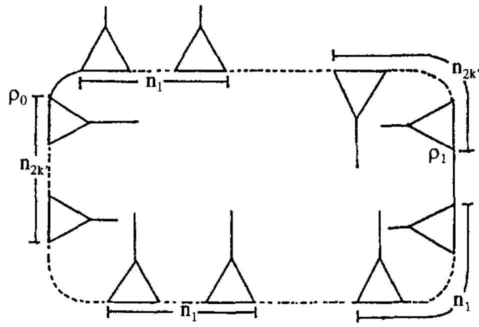

Fig. 2.

and so the circuit will be as shown in Fig. 2. (Note that it does not matter if we reverse the orientation of the triangles on the circuit.)

If $\rho_0, \rho_1, \dots, \rho_{k/k'}$ are the vertices of the triangles on the circuit denoted by heavy dots and $g = (xy)^{n_1}(xy^{-1})^{n_2} \cdots (xy^{-1})^{n_{2k'}} \neq 1$ then $\rho_{i+1} = \rho_i g$ where $i = 0, 1, 2, \dots, (k/k' - 1)$ are indices modulo k/k' and so $\rho_0 \neq \rho_0 g$. Since for all i, $\rho_i = \rho_i(g)^{k/k'}$ and $(g)^{k/k'} \neq 1$, therefore it is a contradiction to the fact that if $g \neq 1$ is an element of G then g has 1 or 2 fixed points and these are the only fixed points of it, unless $g^m = 1$ for some suitable m. Thus, no orbit contains a circuit of the type

$$(n_1, n_2, \dots, n_{2k'}, n_1, n_2, \dots, n_{2k'}, \dots, n_1, n_2, \dots, n_{2k'}). \qquad \square$$

Theorem 2.4. *For a given sequence $n_1, n_2, \dots, n_{2k}$ of positive integers there exists a real quadratic irrational number α such that the circuit in the orbit of α under G has the type $(n_1, n_2, \dots, n_{2k})$ if the circuit does not have a period of even length.*

Proof. A necessary condition, that is, the sequence has no circuit with 'repetitions', has already been established in Theorem 2.3. To show that this condition is sufficient we need only to show that a fixed point v of $g = (xy)^{n_1}(xy^{-1})^{n_2} \cdots (xy^{-1})^{n_{2k}}$ is a real quadratic irrational number.

By Theorem 2.1, g is not a conjugate of x, $y^{\pm 1}$ and $(xy)^n$ because it fixes v and so the trace (calculated by using Theorem 2.2) $r = 2 + \sum_{J \in P} n_J$ where $n_J = \prod_{i \in J} n_i$ of the matrix $M(g)$ is greater than 2, that is, $\sqrt{r^2 - 4}$ is not a complex number. We shall point out that $r^2 - 4$ cannot be a perfect square because otherwise we will be dealing with the coset diagrams in the rational case [2] and so v being a fixed point of g must be ∞ which we know is not the case. Thus, v is a real quadratic irrational number and so belongs to the orbit of α under G. But by Theorem 4 in [5], in a coset diagram for the orbit of α under G the ambiguous numbers form a set of circuits. Hence, by Theorem 2.3 it implies that the orbit contains the circuit of the type $(n_1, n_2, \dots, n_{2k})$. $\square$

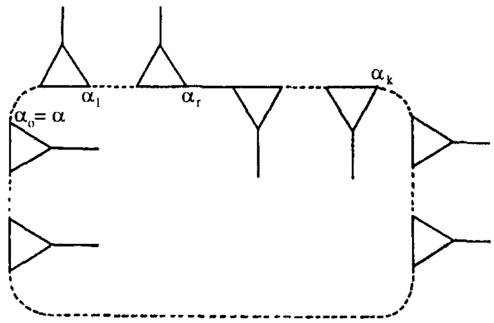

Fig. 3.

Consideration of the action of G on the real quadratic irrational numbers suggests the importance of circuits. In the following theorem we shall give the necessary and sufficient condition for a circuit to contain with real quadratic irrational number α its conjugate $\bar{\alpha}$.

Theorem 2.5. *A circuit contains with α its conjugate $\bar{\alpha}$ if and only if the circuit is of the type $(n_1, n_2, \ldots, n_{k-1}, n_k, n_k, \ldots, n_2, n_1)$.*

Proof. First we note that if α and $\bar{\alpha}$ are conjugates then so are αg and $\bar{\alpha} g$ for every g in G. This means that the theorem is true for every element on the circuit if it is true for any one element.

Let $\alpha, \bar{\alpha}$ belong to the circuit and α be fixed by $g = (xy)^{n_1}(xy^{-1})^{n_2} \cdots (xy^{-1})^{n_{2k}}$, where $n_i > 0$ for $i = 1, 2, \ldots, 2k$. Let us index vertices of the triangles belonging to the circuit as in Fig. 3 by the finite set $\{1, 2, \ldots, m\}$.

If the α's occupy vertices with odd labels then no $\bar{\alpha}$ can occupy any of these vertices. For otherwise $\bar{\alpha} = \alpha(xy)^{n_1}(xy^{-1})^{n_2} \cdots (xy^{-1})^{n_r}$ for some $r < k$ implies that α and $\bar{\alpha}$ are the fixed points of $g = (xy)^{n_1}(xy^{-1})^{n_2} \cdots (xy^{-1})^{n_{2k}}$ and $h = (xg)^{n_{r+1}}(xy^{-1})^{n_{r+2}} \cdots (xy^{-1})^{n_{2k}}(xy)^{n_1}(xy^{-1})^{n_2} \cdots (xy^{-1})^{n_r}$, respectively. Since α and $\bar{\alpha}$ are conjugates, therefore they are fixed by the same element g of G and so g must be equal to h. But this cannot be the case because if it is so then $g = f^s$ for some $s > 1$ and then α will be a fixed point of f. By Theorem 2.3 this cannot happen except for g^t, $t \geqslant 1$ and so gives a contradiction. Thus, $\bar{\alpha}$ cannot occupy an odd vertex. This means that all $\bar{\alpha}$ occupy the vertices which are indexed with even numbers and so α is fixed under the transformation $h = (xy)^{n_k}(xy^{-1})^{n_{k-1}} \cdots (xy)^{n_1}(xy^{-1})^{n_{2k}} \cdots (xy^{-1})^{n_{k+1}}$. This shows that α corresponds to $(n_1, n_2, \ldots, n_{2k})$ and $\bar{\alpha}$ corresponds to $(n_k, n_{k-1}, \ldots, n_1, n_{2k}, n_{2k-1}, \ldots, n_{k+1})$ but now the orientation of the triangles is in the reversed order. That is, the type of the circuit corresponding to α is the same as that corresponding to $\bar{\alpha}$ but with signs reversed and starting at a different point. These types must be the same and so the circuit containing both α and $\bar{\alpha}$ must be of the type $(n_1, n_2, \ldots, n_k, n_k, \ldots, n_2, n_1)$.

3. Action of G on $\mathrm{PL}(F_q)$

Let there exist a circuit of the type $(n_1, n_2, \ldots, n_{2k})$ in a coset diagram representing the action of G on Ω. The following question can be raised. When does a copy of this circuit occur in the coset diagram representing the action of G on $\mathrm{PL}(F_q)$?

If a copy of the circuit occurs in the diagram for the action of G on $\mathrm{PL}(F_q)$ and $g: z \to (az+b)/(cz+d)$ is a non-trivial element of G such that it fixes α, where α belongs to the circuit, then there exists k_0 in $\mathrm{PL}(F_q)$ such that k_0 is fixed by g. That is, k_0 satisfies the quadratic equation $cz^2 + (d-a)z - b \equiv 0 \pmod p$. Since g fixes α, the solutions of the equation are of the form $r + s\sqrt{n}$, where r and s are rational numbers. This implies that $(k_0 - r)/s$ is a square root of n modulo p and so n is a quadratic residue modulo p.

Conversely, as is generally known, there are two distinct projections from Ω to $\mathrm{PL}(F_q)$. The copy of the circuit under either of these projections will be of the type $(n_1, n_2, \ldots, n_{2k})$ provided the projection is one-to-one on the circuit.

In the coset diagram for the action of G on Ω, a point v is on a circuit if and only if it is fixed by some $g = (xy)^{n_1}(xy^{-1})^{n_2} \cdots (xy^{-1})^{n_{2k}}$, where $n_i > 0$ for $i = 1, 2, \ldots, 2k$, of G. This means that the circuits are permuted by any permutation g of Ω which normalises the set $\{xy, xy^{-1}\}$. One such permutation is the field automorphism $s: z \to \bar{z}$ and the other is the transformation $t: z \to 1/z$. Since $s^2 = t^2 = (st)^2 = 1$, we have 4-permutation group permuting the circuits. Thus, we ask: for positive integers $n_1, n_2, \ldots, n_{2k}$ what is the orbit under $V_4 = \langle s, t : s^2 = t^2 = (st)^2 = 1 \rangle$ containing the circuit of the type $(n_1, n_2, \ldots, n_{2k})$? Specifically, it means under what condition(s) the circuit of the type $(n_1, n_2, \ldots, n_{2k})$ contains

(i) with α its image $\bar{\alpha}$ under s;
(ii) with α its image $1/\alpha$ under t;
(iii) with α its image $1/\bar{\alpha}$ under st?

Theorem 2.5 answers our first question. To answer questions (ii) and (iii) we proceed as follows.

Theorem 3.1. *For given positive integers $n_1, n_2, \ldots, n_{2k}$ the circuit in the coset diagram for the action of G on $\mathrm{PL}(F_q)$ contains with α its image β under $t: z \to 1/z$ if and only if the circuit is of the type $(n_1, n_2, \ldots, n_k, n_1, n_2, \ldots, n_k)$.*

Proof. Suppose the circuit is of the type $(n_1, n_2, \ldots, n_{2k})$ and it contains with α its image $\beta = (\alpha)t$. Let $\alpha = \alpha_0$ be the vertex fixed by $g = (xy)^{n_1}(xy^{-1})^{n_2} \cdots (xy^{-1})^{n_{2k}}$, where $n_i > 0$ for all $i = 1, 2, \ldots, 2k$. We index vertices of the triangles in Fig. 4.

We note that no β can occupy any of the vertices occupied by α, for if otherwise then $\beta = \alpha_r$ for some $r < k$. This implies that β is fixed by

$$g = (xy^{-1})^{n_{r+1}}(xy)^{n_{r+2}} \cdots (xy^{-1})^{n_{2k}}(xy)^{n_1} \cdots (xy)^{n_r}$$

and α is fixed by

$$h = (xy)^{n_1}(xy^{-1})^{n_2} \cdots (xy)^{n_r}(xy^{-1})^{n_{r+1}} \cdots (xy^{-1})^{n_{2k}}.$$

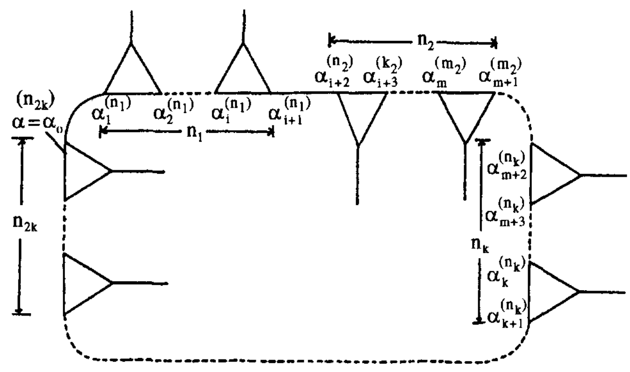

Fig. 4.

But $g = h$ for otherwise $g = f^l$ for some $l > 1$ which we know due to Theorem 2.4, cannot happen. Thus, β cannot occupy any of these vertices.

The quadratic forms of $\alpha, \bar{\alpha}$ and $\beta, \bar{\beta}$ are, respectively, $q_1 : cx^2 - 2axy + by^2$ and $q_2 : bx^2 - 2axy + cy^2$ and so the corresponding symmetric matrices with the same trace are

$$\begin{bmatrix} c & -a \\ -a & b \end{bmatrix} \text{ and } \begin{bmatrix} b & -a \\ -a & c \end{bmatrix}.$$

This shows that the element which fixes α and β should be, respectively,

$$g = (xy)^{n_1}(xy^{-1})^{n_2} \cdots (xy^{-1})^{n_{2k}} \quad \text{and}$$

$$h = (xy^{-1})^{n_1}(xy)^{n_2} \cdots (xy)^{n_{2k}}.$$

This shows that half of the circuit repeats but with orientations of the triangles on the circuit are all reversed. That is, $(n_1, n_2, \ldots, n_{2k})$ form the same circuit after the reflection. Hence the result.

Theorem 3.2. *For given positive integers $n_1, n_2, \ldots, n_{2k}$ the circuit in the coset diagram for the action of G on $\mathrm{PL}(F_q)$ contains with α its image β under the transformation $st : z \to 1/\bar{z}$ if and only if the circuit is of the type $(n_1, n_2, \ldots, n_k, n_{k+1}, n_k, \ldots, n_2)$.*

Proof. Let the vertex α, belonging to $(n_1, n_2, \ldots, n_k)$, be a fixed point of $g = (xy)^{n_1}(xy^{-1})^{n_2} \cdots (xy^{-1})^{n_{2k}}$. Since $\alpha x = \alpha(xy)^{n_1}(xy^{-1})^{n_2} \cdots (xy^{-1})^{n_{2k}}x = \alpha x(yx)^{n_1}(yx^{-1})^{n_2} \cdots (yx)^{n_{2k}} = \alpha x(xy)^{n_{2k}}(xy^{-1})^{n_{2k-1}} \cdots (xy^{-1})^{n_1}$, therefore $t : z \to 1/z$ reflects $(n_1, n_2, \ldots, n_{2k})$ and induces the circuit of the type $(n_1, n_{2k}, \ldots, n_2)$. This means that the vertex αt is fixed by the element $h = (xy)^{n_1}(xy^{-1})^{n_{2k}} \cdots (xy^{-1})^{n_2}$. If the circuit of the type $(n_1, n_2, \ldots, n_{2k})$ contains with α its conjugate $\bar{\alpha}$ then $\bar{\alpha}$ is fixed by $g' = (xy^{-1})^{n_1}(xy)^{n_2} \cdots (xy)^{n_{2k}}$ and so because of t the reflected circuit $(n_1, n_{2k}, \ldots, n_2)$

contains with αt its conjugate $\bar{\alpha}t$ and that $\bar{\alpha}t$ is fixed by $h'(xy^{-1})^{n_{2i}}(xy)^{n_{2i+1}} \cdots (xy)^{n_{2k-2}} (xy^{-1})^{n_{2k-1}}(xy)^{n_{2k}} \cdots (xy)^{n_{2i-1}}$. Since h and h' must be equal, therefore $(n_1, n_{2k}, \dots, n_2)$ must coincide with even cyclic permutation $(n_{2i}, n_{2i+1}, \dots, n_{2k-2}, n_{2k-1}, n_{2k}, \dots, n_{2i-1})$ but now the orientation of all the triangles is reversed. We note that h' is of the same length as h, for otherwise h being a power of a shorter circuit of even length will give a shorter circuit. Since $(n_1, n_2, \dots, n_{2k})$ contains αt and $\bar{\alpha}t$, by Theorem 2.5, it must coincide with $(n_1, n_2, \dots, n_k, n_k, \dots, n_1)$. Thus, $(n_1, n_{2k}, \dots, n_2) = (n_{2i}, n_{2i+1}, \dots, n_{2k-2}, n_{2k-1}, \dots, n_{2i-1}) = (n_1, n_2, \dots, n_k, n_k, \dots, n_1)$ implies that the circuit will be of the type $(n_1, n_2, \dots, n_{k+1}, n_k, \dots, n_2)$.

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